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**EQUATIONS FOR CORRELATION FUNCTIONS
OF EIGHT-VERTEX MODEL:
FERROMAGNETIC AND DISORDERED PHASES***

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The Kyoto group (Jimbo, Miwa, Nakayashiki *et al.*) showed that the partition function and correlation functions of the eight-vertex model in antiferromagnetic phases can be calculated using simple analytical properties of the R -matrix. We extend these methods to ferromagnetic and disordered phases. We use Baxter's symmetries to obtain appropriate parametrizations of the R -matrix and to substantiate the validity of the analytical approach for these phases. These symmetries allow one to relate correlation functions in different phases.

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1. Introduction

Exactly solvable lattice models have numerous applications to statistical physics, quantum field theory and solid state physics. Effective methods for calculating partition functions based on analytical properties were proposed by Baxter.¹ Recently, the Kyoto group (Jimbo, Miwa, Nakayashiki, ...) considerably simplified these methods and developed an approach to calculating correlation functions at finite distances.²⁻⁸ They applied this approach to the antiferromagnetic phases of the six-vertex^{2,3} and eight-vertex^{7,8} models and to the SOS and RSOS models.⁴⁻⁶ An interesting challenge is to generalize this approach to ferromagnetic and disordered phases of the eight-vertex (and six-vertex) model. The problem is that the usual arguments for applicability of the corner transfer matrix approach break down in these phases. On the other hand, Baxter found some symmetries between different phases of the eight-vertex model.¹ More precisely, the partition function is invariant under some transformations of local weights, and these transformations connect values of weights corresponding to different stable phases. We show that these symmetries can be extended to correlation functions. Then one can express correlation functions in ferromagnetic and disordered phases in terms of correlators in the antiferromagnetic phase. We formulate the rules for calculating correlation functions directly in the ferromagnetic and disordered phases using appropriate parametrizations of the local weights.

Now we state the notations. ‘Spins’ of the eight-vertex model $\varepsilon = \pm \equiv \pm 1$ are situated at the links. The interaction is attached to the vertices and described by the weight matrix (or R -matrix) $R_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4}$ (Fig. 1a)

$$R = \begin{array}{cc} & \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array} & \left(\begin{array}{cccc} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{array} \right) \end{array}. \quad (1.1)$$

The partition function is given by (see Fig. 2)

$$Z = \sum_{\{\mu_{kl}, \nu_{kl}\}} \prod_{k,l \in \mathbf{Z}} R_{\mu_{k-1,l}, \nu_{kl}}^{\mu_{kl}, \nu_{k,l-1}}.$$

We need one more element (Fig. 1b). It is the spin flop operator $\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. It may be considered as a kind of a two-link vertex.

$$R_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} = \begin{array}{c} \varepsilon_3 \\ | \\ \varepsilon_2 \text{---} \varepsilon_4 \\ | \\ \varepsilon_1 \end{array} \quad (\sigma^1)_{\varepsilon_1}^{\varepsilon_2} = \begin{array}{c} \varepsilon_1 \text{---} \varepsilon_2 \\ \times \end{array}$$

(a)
(b)

Fig. 1

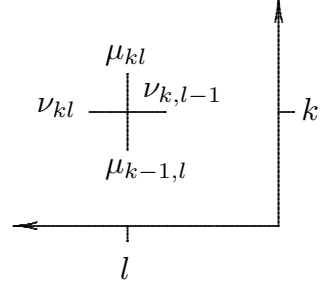


Fig. 2

There are five regions of parameters¹ a, b, c, d

$$\begin{aligned}
 A_1 : & \quad c > a + b + d \quad (\text{main region}), \\
 A_2 : & \quad d > a + b + c, \\
 D : & \quad \frac{1}{2}(a + b + c + d) > a, b, c, d, \\
 F_1 : & \quad a > b + c + d, \\
 F_2 : & \quad b > a + c + d.
 \end{aligned} \tag{1.2}$$

The regions A_1 and A_2 are antiferromagnetic, F_1 and F_2 ferromagnetic, and D disordered. The case $d = 0$ corresponds to the six-vertex model.

The solution of the eight-vertex model in the antiferromagnetic region is based on appropriate parametrization of the R -matrix so that it satisfies some functional equations. We recall main points of this approach in Sec. 2. A simple local transformation makes it possible to find necessary parametrization in the ferromagnetic case (Sec. 3). Some nonlocal transformation (‘duality’) connects the ferromagnetic region with the disordered one (Sec. 4). A particular case of the six-vertex model is considered in Sec. 5. Several concluding remarks are given in Sec. 6.

2. Antiferromagnetic Region

In this section we recall main points of the Kyoto approach. Consider the antiferromagnetic region A_1 . Let $a_1^{(i)}$, $i = 0, 1$ be ground state configurations

$$\mu_{kl} = \nu_{kl} = (-)^{k+l+i}. \tag{2.1}$$

Let $A_1^{(i)}$ be the set of configurations that differ from the ground state $a_1^{(i)}$ by a finite number of spin flops. Only configurations from $A_1^{(0)} \cup A_1^{(1)}$ contribute to the partition function.

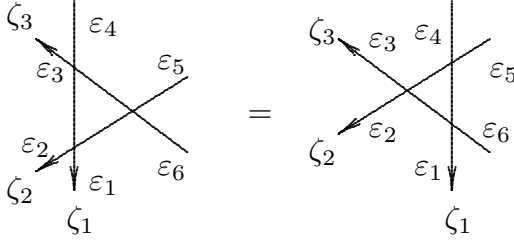


Fig. 3

The R -matrix can be parametrized as follows^{1,7}

$$\begin{aligned}
a(\zeta) &\equiv a(\zeta|q, x) = \rho(\zeta) \cdot \frac{\zeta}{x} \Theta_{q^2}(qx^2) \Theta_{q^2}(q\zeta^2) \Theta_{q^2}(x^2/\zeta^2), \\
b(\zeta) &\equiv b(\zeta|q, x) = \rho(\zeta) \cdot \frac{1}{\zeta} \Theta_{q^2}(qx^2) \Theta_{q^2}(\zeta^2) \Theta_{q^2}(qx^2/\zeta^2), \\
c(\zeta) &\equiv c(\zeta|q, x) = \rho(\zeta) \cdot \frac{1}{x} \Theta_{q^2}(x^2) \Theta_{q^2}(q\zeta^2) \Theta_{q^2}(qx^2/\zeta^2), \\
d(\zeta) &\equiv d(\zeta|q, x) = \rho(\zeta) \cdot \frac{q^{\frac{1}{2}}}{x^2} \Theta_{q^2}(x^2) \Theta_{q^2}(\zeta^2) \Theta_{q^2}(x^2/\zeta^2).
\end{aligned} \tag{2.2}$$

Here

$$\begin{aligned}
\Theta_p(z) &= (z; p)_\infty (p/z; p)_\infty (p; p)_\infty, \\
(z; p_1, \dots, p_N)_\infty &= \prod_{n_1, \dots, n_N=0}^{\infty} (1 - zp_1^{n_1} \dots p_N^{n_N}).
\end{aligned}$$

The function $\Theta_p(z)$ is connected with the standard theta-function $\Theta(u)$ as follows

$$\Theta(iu) = \Theta_{q^2} \left(qe^{-\pi u/I} \right), \quad q = e^{-\pi I'/I},$$

with I, I' being the standard halfperiods.

The region A_1 corresponds to the values of parameters

$$0 < q^{\frac{1}{2}} < x < \zeta < 1. \tag{2.3}$$

The limit $q \rightarrow 0$ corresponds to the six-vertex model.

The R -matrix satisfies the Yang–Baxter equation¹

$$\begin{aligned}
\sum_{\alpha\beta\gamma} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2}^{\alpha\beta} R(\zeta_1/\zeta_3)_{\alpha\varepsilon_3}^{\varepsilon_4\gamma} R(\zeta_2/\zeta_3)_{\beta\gamma}^{\varepsilon_5\varepsilon_6} \\
= \sum_{\alpha'\beta'\gamma'} R(\zeta_2/\zeta_3)_{\varepsilon_2\varepsilon_3}^{\beta'\gamma'} R(\zeta_1/\zeta_3)_{\varepsilon_1\gamma'}^{\alpha'\varepsilon_6} R(\zeta_1/\zeta_2)_{\alpha'\beta'}^{\varepsilon_4\varepsilon_5}.
\end{aligned} \tag{2.4}$$

Grafically this equation is presented in Fig. 3. Spectral parameters ζ_i are attached to oriented lines.

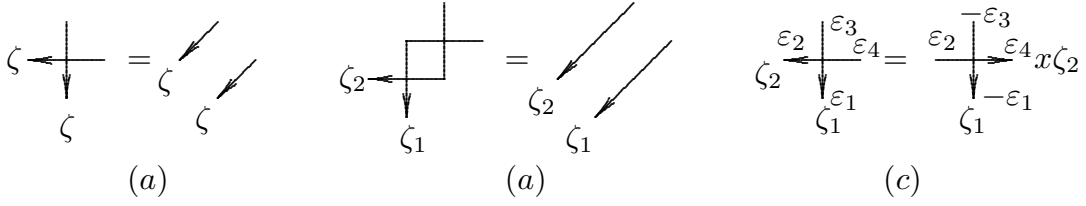


Fig. 4

The function $\rho(\zeta)$ in Eq. (2.2) is chosen so that the partition function ‘per cite’ is equal to unity. It is equivalent to the conditions⁷ (Fig. 4)

$$R(1)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} = \delta_{\varepsilon_1}^{\varepsilon_4} \delta_{\varepsilon_2}^{\varepsilon_3} \quad (\text{initial condition}), \quad (2.5a)$$

$$\sum_{\alpha \beta} R(\zeta)_{\varepsilon_1 \varepsilon_2}^{\alpha \beta} R(\zeta^{-1})_{\beta \alpha}^{\varepsilon_3 \varepsilon_4} = \delta_{\varepsilon_1}^{\varepsilon_4} \delta_{\varepsilon_2}^{\varepsilon_3} \quad (\text{unitarity}), \quad (2.5b)$$

$$R(\zeta)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} = R(x/\zeta)_{\varepsilon_4, -\varepsilon_1}^{\varepsilon_2, -\varepsilon_3} \quad (\text{crossing symmetry}). \quad (2.5c)$$

The last condition (2.5c) is in a sense the crucial one. It is not connected with normalization. Unnormalized but obviously analytical R -matrix $\rho^{-1}(\zeta) R(\zeta)$ satisfies Eq. (2.5c). Therefore corresponding partition function ‘per cite’ $\rho^{-1}(\zeta)$ satisfies the equation

$$\rho(x/\zeta) = \rho(\zeta). \quad (2.5c')$$

It should be mentioned that in the region A_1 the crossing (2.5c) does not change boundary conditions (2.1). Moreover, going from ζ in (2.3) to x/ζ we do not intersect the surfaces of phase transitions. Indeed, if $0 < x < \zeta < 1$ then $0 < x < x/\zeta < 1$. But within one region the partition function per cite $\rho^{-1}(\zeta)$ is analytical and the condition (2.5c') is valid as an equation for an analytical function $\rho(\zeta)$. It is just Eq. (2.5c) for the parametrization (2.2) that breaks down in the ferromagnetic and disordered regions. Indeed, in these regions the right hand side and the left hand side correspond to different phases with different greatest eigenvalues of the transfer matrix. The partition function per cite $\rho^{-1}(\zeta)$ for unnormalized analytical weights cannot be analytically continued over the phase transition. Hence, the parametrization (2.2) is useless in ferromagnetic and disordered phases.

The solution to Eqs. (2.5) is given by^{1,7}

$$\begin{aligned} \rho(\zeta) &= x(q; q)_{\infty}^{-2} (q^2; q^2)_{\infty}^{-1} (x^2 \zeta^{-2}; q)_{\infty}^{-1} (qx^{-2} \zeta^2; q)_{\infty}^{-1} \\ &\times \frac{(x^4 \zeta^2; q, x^4)_{\infty} (x^2 \zeta^{-2}; q, x^4)_{\infty} (q \zeta^2; q, x^4)_{\infty} (qx^2 \zeta^{-2}; q, x^4)_{\infty}}{(x^4 \zeta^{-2}; q, x^4)_{\infty} (x^2 \zeta^2; q, x^4)_{\infty} (q \zeta^{-2}; q, x^4)_{\infty} (qx^2 \zeta^2; q, x^4)_{\infty}}. \end{aligned} \quad (2.6)$$

We now turn to correlation functions. The Kyoto approach uses two main types of objects: corner transfer matrices (CTMs) and vertex operators (VOs). The north-west corner transfer matrix,¹ $C_{NW}^{(i)}(\zeta)_{\varepsilon_1 \varepsilon_2' \dots}^{\varepsilon_1' \varepsilon_2' \dots}$, is the partition function

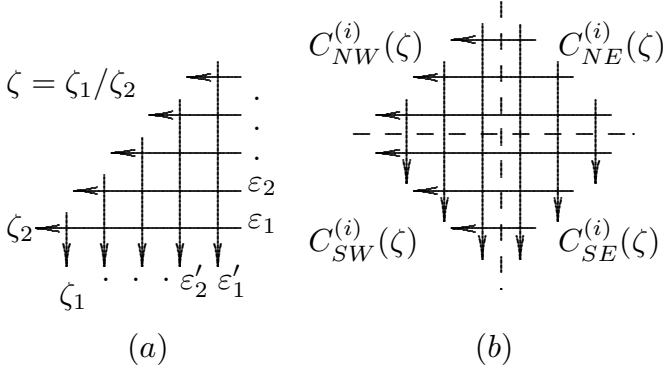


Fig. 5

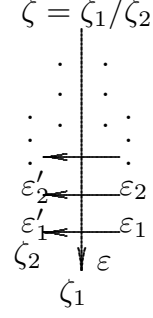


Fig. 6

on the north-west quadrant over the set $A_1^{(i)}$ with the configuration $\varepsilon_1 \varepsilon_2 \dots$ on the vertical boundary and $\varepsilon'_1 \varepsilon'_2 \dots$ on horizontal one (Fig. 5a). The definitions of the south-west, south-east and north-east CTMs, $C_{SW}^{(i)}(\zeta)$, $C_{SE}^{(i)}(\zeta)$ and $C_{NE}^{(i)}(\zeta)$, (Fig. 5b) are evident. On the infinite lattice

$$C_{NW}^{(i)}(\zeta) = \zeta^{D^{(i)}}, \quad (2.7a)$$

$$C_{SW}^{(i)}(\zeta) = \sigma_\infty^1 (x/\zeta)^{D^{(i)}}, \quad (2.7b)$$

$$C_{SE}^{(i)}(\zeta) = \sigma_\infty^1 \zeta^{D^{(i)}} \sigma_\infty^1, \quad (2.7c)$$

$$C_{NE}^{(i)}(\zeta) = (x/\zeta)^{D^{(i)}} \sigma_\infty^1, \quad (2.7d)$$

$$C_{NE}^{(i)}(\zeta) C_{SE}^{(i)}(\zeta) C_{SW}^{(i)}(\zeta) C_{NW}^{(i)}(\zeta) = x^{2D^{(i)}}, \quad (2.7e)$$

where $D^{(i)}$ is an operator independent of ζ , and

$$\sigma_\infty^1 = \sigma^1 \otimes \sigma^1 \otimes \dots$$

is a spin flop operator along one boundary of a CTM. Note that Eqs. (2.7b–d) follow from Eq. (2.7a) and the crossing symmetry (2.5c). It means that they only hold for the antiferromagnetic region.

The vertex operator

$$\Phi_\varepsilon^{(1-i,i)}(\zeta)_{\varepsilon_1 \varepsilon_2 \dots}^{\varepsilon'_1 \varepsilon'_2 \dots} = \sum_{\{\mu_k\}} \prod_{k=1}^{\infty} R(\zeta)_{\mu_{k-1} \varepsilon'_k}^{\mu_k \varepsilon_k} \Big|_{\mu_0 = \varepsilon}, \quad \varepsilon_k = -\varepsilon'_k = (-)^{k+i+1}, \quad k \gg 0. \quad (2.8)$$

is a partition function along a half line (Fig. 6). VOs satisfy the equations

$$\sum_{\varepsilon'_1 \varepsilon'_2} R(\zeta_1/\zeta_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} \Phi_{\varepsilon'_1}^{(1-i,i)}(\zeta_1) \Phi_{\varepsilon'_2}^{(i,1-i)}(\zeta_2) = \Phi_{\varepsilon_2}^{(1-i,i)}(\zeta_2) \Phi_{\varepsilon_1}^{(i,1-i)}(\zeta_1), \quad (2.9a)$$

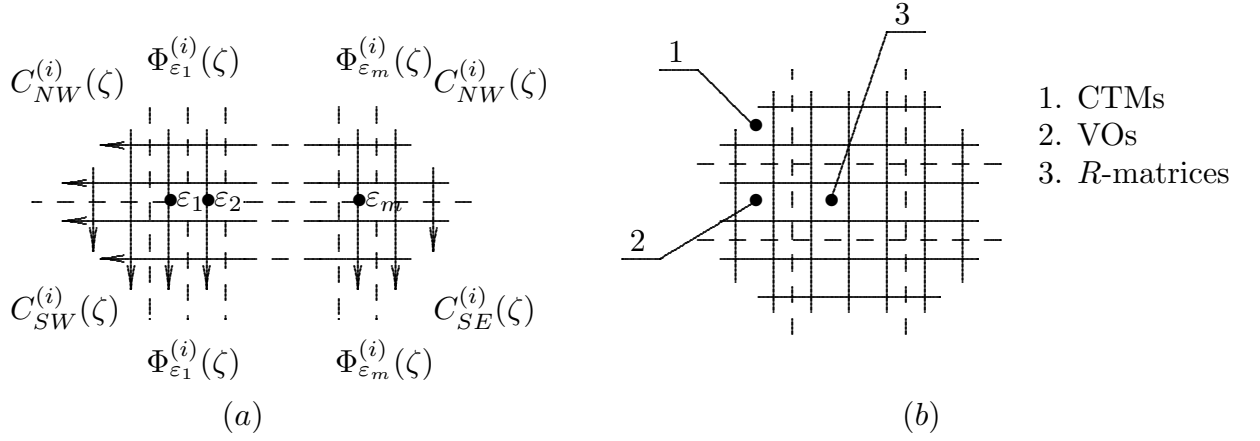


Fig. 7

$$\xi^{D^{(1-i)}} \Phi_{\varepsilon}^{(1-i,i)}(\zeta) = \Phi_{\varepsilon}^{(1-i,i)}(\zeta/\xi) \xi^{D^{(i)}}, \quad (2.9b)$$

$$\sigma_{\infty}^1 \Phi_{\varepsilon}^{(1-i,i)}(\zeta) = \Phi_{-\varepsilon}^{(i,1-i)}(\zeta) \sigma_{\infty}^1. \quad (2.9c)$$

Two first equations follow from the Yang–Baxter equation (2.4), Eq. (2.9c) is evident.

Every correlation function can be expressed in terms of vertex operator correlation functions

$$F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n) = \text{Tr} \left(x^{2D^{(i)}} \Phi_{\varepsilon_1}^{(i,1-i)}(\zeta_1) \dots \Phi_{\varepsilon_n}^{(1-i,i)}(\zeta_n) \right), \quad n = 0, 2, 4, \dots. \quad (2.10)$$

Indeed, consider, for example, the probability $P_{\varepsilon_1 \dots \varepsilon_m}$ that m spins along a row of parallel links take the values $\varepsilon_1, \dots, \varepsilon_m$ (Fig. 7a) in the phase $A_1^{(i)}$. Then

$$\begin{aligned} P_{\varepsilon_1 \dots \varepsilon_m} &= \text{Tr} \left(C_{NE}^{(i')}(\zeta) C_{SE}^{(i')}(\zeta) \Phi_{\varepsilon_m}^{(i',1-i')}(\zeta) \dots \Phi_{\varepsilon_1}^{(1-i,i)}(\zeta) \right. \\ &\quad \times C_{SW}^{(i)}(\zeta) C_{NW}^{(i)}(\zeta) \Phi_{\varepsilon_1}^{(i,1-i)}(\zeta) \dots \Phi_{\varepsilon_m}^{(1-i',i')}(\zeta) \left. \right) \\ &= F_{-\varepsilon_m \dots -\varepsilon_1 \varepsilon_1 \dots \varepsilon_m}(\underbrace{x\zeta, \dots, x\zeta}_m, \underbrace{\zeta, \dots, \zeta}_m). \end{aligned} \quad (2.11)$$

The last line is obtained using Eqs. (2.9b) and (2.9c). In general case it is necessary to divide the crystal (Fig. 7b) into the CTMs 1, products of VOs 2, and a finite cluster of R -matrices 3.

VO correlation functions can be found by solving the equations⁷

$$\begin{aligned}
F_{\varepsilon_1 \dots \varepsilon_n}^{(i)}(\xi \zeta_1, \dots, \xi \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}^{(i)}(\zeta_1, \dots, \zeta_n), \\
\sum_{\varepsilon'_j \varepsilon'_{j+1}} R(\zeta_j / \zeta_{j+1})_{\varepsilon_j \varepsilon_{j+1}}^{\varepsilon'_j \varepsilon'_{j+1}} F_{\dots \varepsilon'_j \varepsilon'_{j+1} \dots}^{(i)}(\dots, \zeta_j, \zeta_{j+1}, \dots) &= F_{\dots \varepsilon_{j+1} \varepsilon_j \dots}^{(i)}(\dots, \zeta_{j+1}, \zeta_j, \dots), \\
F_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}^{(i)}(x^2 \zeta_1, \zeta_2, \dots, \zeta_n) &= F_{\varepsilon_2 \dots \varepsilon_n \varepsilon_1}^{(1-i)}(\zeta_2, \dots, \zeta_n, \zeta_1), \\
\sum_{\varepsilon} F_{-\varepsilon \varepsilon \varepsilon_1 \dots \varepsilon_n}^{(i)}(x \zeta, \zeta, \zeta_1, \dots, \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}^{(i)}(\zeta_1, \dots, \zeta_n), \\
F^{(i)}() &= 1.
\end{aligned} \tag{2.12}$$

For example,⁷

$$\frac{\sum_{\varepsilon} \varepsilon F_{-\varepsilon \varepsilon}^{(0)}(\zeta, 1)}{\sum_{\varepsilon} F_{-\varepsilon \varepsilon}^{(0)}(\zeta, 1)} = \frac{(x^3 \zeta^{-1}; x^2)_{\infty} (x \zeta; x^2)_{\infty} (-x \zeta^{-1}; q)_{\infty} (-q x^{-1} \zeta; q)_{\infty}}{(-x^3 \zeta^{-1}; x^2)_{\infty} (-x \zeta; x^2)_{\infty} (x \zeta^{-1}; q)_{\infty} (q x^{-1} \zeta; q)_{\infty}},$$

and the Baxter–Kelland formula¹ for the ‘staggered’ spontaneous polarization holds

$$\langle \varepsilon \rangle = \frac{(x^2; x^2)_{\infty}^2 (-q; q)_{\infty}^2}{(-x^2; x^2)_{\infty}^2 (q; q)_{\infty}^2}. \tag{2.13}$$

3. Ferromagnetic Region

Consider the eight-vertex model in the main region A_1 . Consider the substitution

$$\mu_{kl} = (-)^{k+l} \mu'_{kl}, \quad \nu_{kl} = (-)^{k+l} \nu'_{kl}. \tag{3.1}$$

Introduce new R -matrix, R_F , so that

$$R_{\mu_{k-1,l} \nu_{kl}}^{\mu_{kl} \nu_{k,l-1}} = (R'_F)_{\mu'_{k-1,l} \nu'_{kl}}^{\mu'_{kl} \nu'_{k,l-1}} = (R'_F)_{\mu_{k-1,l} \nu_{kl}}^{-\mu_{kl} \nu_{k,l-1}}, \tag{3.2a}$$

or, in components,

$$a_F = c, \quad b_F = d, \quad c_F = a, \quad d_F = b. \tag{3.2b}$$

Evidently

$$a_F > b_F, c_F, d_F,$$

and the matrix R_F describes the model in the ferromagnetic F_1 region. The ground state antiferromagnetic configuration $a_1^{(i)}$ transforms to a ground state ferromagnetic configuration $f_1^{(i)}$

$$\mu_{kl} = \nu_{kl} = (-)^i \tag{3.3}$$

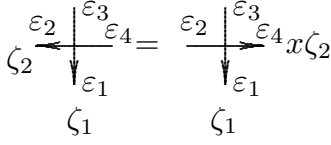


Fig. 8

under the substitution (3.1). Similarly, the set $A_1^{(i)}$ transforms to the set $F_1^{(i)}$ of configurations that differ from $f_1^{(i)}$ by finite number of spin flops.

The substitution (3.1) defines a natural parametrization in the ferromagnetic region

$$\begin{aligned}
a_F(\zeta) &= \rho(\zeta) \cdot \frac{1}{x} \Theta_{q^2}(x^2) \Theta_{q^2}(q\zeta^2) \Theta_{q^2}(qx^2/\zeta^2), \\
b_F(\zeta) &= \rho(\zeta) \cdot \frac{q^{\frac{1}{2}}}{x^2} \Theta_{q^2}(x^2) \Theta_{q^2}(\zeta^2) \Theta_{q^2}(x^2/\zeta^2), \\
c_F(\zeta) &= \rho(\zeta) \cdot \frac{\zeta}{x} \Theta_{q^2}(qx^2) \Theta_{q^2}(q\zeta^2) \Theta_{q^2}(x^2/\zeta^2), \\
d_F(\zeta) &= \rho(\zeta) \cdot \frac{1}{\zeta} \Theta_{q^2}(qx^2) \Theta_{q^2}(\zeta^2) \Theta_{q^2}(qx^2/\zeta^2),
\end{aligned} \tag{3.4}$$

with q , x and ζ in the region (2.3). The R -matrix $R_F(\zeta)$ satisfies the Yang–Baxter equation (2.4) and equations (2.5a,b). The crossing symmetry is different (Fig. 8)

$$R_F(\zeta)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} = R_F(x/\zeta)_{\varepsilon_4 \varepsilon_1}^{\varepsilon_2 \varepsilon_3}. \tag{3.5}$$

Then the CTMs are given by

$$\begin{aligned}
C_{NW}^{(i)}(\zeta) &= C_{SE}^{(i)}(\zeta) = \zeta^{D^{(i)}}, \\
C_{SW}^{(i)} &= C_{NE}^{(i)}(\zeta) = (x/\zeta)^{D^{(i)}}
\end{aligned} \tag{3.6}$$

with some new $D^{(i)}$. The VOs, $\Phi_\varepsilon^{(i)}(\zeta)$ satisfy the equations

$$\begin{aligned}
\sum_{\varepsilon'_1 \varepsilon'_2} R_F(\zeta_1/\zeta_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} \Phi_{\varepsilon'_1}^{(i)}(\zeta_1) \Phi_{\varepsilon'_2}^{(i)}(\zeta_2) &= \Phi_{\varepsilon_2}^{(i)}(\zeta_2) \Phi_{\varepsilon_1}^{(i)}(\zeta_1), \\
\xi^{D^{(i)}} \Phi_\varepsilon^{(i)}(\zeta) &= \Phi_\varepsilon^{(i)}(\zeta/\xi) \xi^{D^{(i)}}.
\end{aligned} \tag{3.7}$$

These equations do not relate objects with different values of i and we omit the superscript $^{(i)}$ from now on if it does not lead to a confusion. The VO correlation functions

$$F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n) = \text{Tr} \left(x^{2D} \Phi_{\varepsilon_1}(\zeta_1) \dots \Phi_{\varepsilon_n}(\zeta_n) \right), \quad n = 0, 1, 2, \dots, \tag{3.8}$$

satisfy the equations

$$\begin{aligned}
F_{\varepsilon_1 \dots \varepsilon_n}(\xi \zeta_1, \dots, \xi \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n), \\
\sum_{\varepsilon'_j \varepsilon'_{j+1}} R_F(\zeta_j / \zeta_{j+1})_{\varepsilon_j \varepsilon_{j+1}}^{\varepsilon'_j \varepsilon'_{j+1}} F_{\dots \varepsilon'_j \varepsilon'_{j+1} \dots}(\dots, \zeta_j, \zeta_{j+1}, \dots) &= F_{\dots \varepsilon_{j+1} \varepsilon_j \dots}(\dots, \zeta_{j+1}, \zeta_j, \dots), \\
F_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}^{(i)}(x^2 \zeta_1, \zeta_2, \dots, \zeta_n) &= F_{\varepsilon_2 \dots \varepsilon_n \varepsilon_1}^{(1-i)}(\zeta_2, \dots, \zeta_n, \zeta_1), \\
\sum_{\varepsilon} F_{\varepsilon \varepsilon \varepsilon_1 \dots \varepsilon_n}^{(i)}(x \zeta, \zeta, \zeta_1, \dots, \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}^{(i)}(\zeta_1, \dots, \zeta_n), \\
F() &= 1, \quad F_{\varepsilon}^{(i)}(\zeta) = \delta_{\varepsilon, (-1)^i}.
\end{aligned} \tag{3.9}$$

The probabilities $P_{\varepsilon_1 \dots \varepsilon_m}$ are given by

$$P_{\varepsilon_1 \dots \varepsilon_m} = F_{\varepsilon_m \dots \varepsilon_1 \varepsilon_1 \dots \varepsilon_m}(\underbrace{x \zeta, \dots, x \zeta}_m, \underbrace{\zeta, \dots, \zeta}_m). \tag{3.10}$$

The substitution (3.1) identifies Eqs. (3.9), (3.10) with Eqs. (2.12), (2.11) for even n . It proves the correctness of the above reasoning. In particular

$$F_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_{n-1} \varepsilon_n}^{(i) \text{ fer.}}(\zeta_1, \zeta_2, \dots, \zeta_{n-1}, \zeta_n) = F_{-\varepsilon_1 \varepsilon_2 \dots -\varepsilon_{n-1} \varepsilon_n}^{(i) \text{ antifer.}}(\zeta_1, \zeta_2, \dots, \zeta_{n-1}, \zeta_n). \tag{3.12}$$

To conclude this section we shall make several remarks.

Firstly, if one does not want to work with equations for VO correlation functions, one may use the parametrization (2.2), but with new value of the parameter x and new function $\rho(\zeta)$. The connection between the parametrizations (2.2) and (3.4) is given in the Appendix.

The six-vertex limit $d_F \rightarrow 0$ corresponds [in parametrization (3.4)!] to

$$q \rightarrow 0, \quad -q^{-\frac{1}{2}} x = \text{const} = x_F. \tag{3.12}$$

Here x_F is the parameter of the six-vertex model corresponding to x in the parametrization (2.2).

The spontaneous polarization in the ferromagnetic phase is given by the same formula (2.13) (but with the relation (3.4) between ζ , q , x and weights!). In the six-vertex limit (3.12) we obtain $\langle \varepsilon \rangle = 1$. This result is consistent with the wellknown fact that the ferromagnetic phase in the six-vertex model is frozen.

Another interesting question is the sense of VO correlation functions with odd n . Physically they correspond to a crystal with dislocations. It is easy to check, for example, that the spin at the link of a single dislocation for the eight-vertex model in the ferromagnetic phase is frozen $\langle \varepsilon \rangle = 1$. There is an analogue to these correlation functions in the antiferromagnetic region: correlation functions with one insertion of σ_{∞}^1 .

4. Disordered Region

The duality transformation¹

$$\begin{aligned}
a_D &= \frac{1}{2}(a_F + b_F + c_F + d_F), \\
b_D &= \frac{1}{2}(a_F + b_F - c_F - d_F), \\
c_D &= \frac{1}{2}(a_F - b_F + c_F - d_F), \\
d_D &= \frac{1}{2}(a_F - b_F - c_F + d_F)
\end{aligned} \tag{4.1}$$

connects the disordered region with the ferromagnetic region F_1 . Physically the duality is based on the representation of the R -matrix

$$\begin{aligned}
(R_F)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= (R_1)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} + \cdots + (R_8)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4}, \\
(R_1)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}a_D, & (R_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}a_D \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4, \\
(R_3)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}b_D \varepsilon_1 \varepsilon_3, & (R_4)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}b_D \varepsilon_2 \varepsilon_4, \\
(R_5)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}c_D \varepsilon_1 \varepsilon_4, & (R_6)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}c_D \varepsilon_2 \varepsilon_3, \\
(R_7)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}d_D \varepsilon_3 \varepsilon_4, & (R_8)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} &= \frac{1}{2}d_D \varepsilon_1 \varepsilon_2.
\end{aligned} \tag{4.2}$$

The partition function can be written as

$$Z = \sum_{\{t_{kl}=1, \dots, 8\}} \sum_{\{\mu_{kl}, \nu_{kl}\}} \prod_{kl} (R_{t_{kl}})_{\mu_{k-1,l}}^{\mu_{kl}} \frac{\nu_{k,l-1}}{\nu_{kl}}.$$

Let us sum up over all μ_{kl} , ν_{kl} . A nonzero contribution into the sum over, for example, μ_{kl} is given by the terms

$$\sum_{\mu_{kl}} (R_{t_{kl}})_{\mu_{k-1,l}}^{\mu_{kl}} \frac{\nu_{k,l-1}}{\nu_{kl}} (R_{t_{k+1,l}})_{\mu_{kl}}^{\mu_{k+1,l}} \frac{\nu_{k+1,l-1}}{\nu_{k+1,l}} \tag{4.3}$$

with such t_{kl} and $t_{k+1,l}$ that the power of μ_{kl} is even. Now one can associate a dual vertex $\begin{smallmatrix} \varepsilon_3 & \varepsilon_4 \\ \varepsilon_1 & \varepsilon_2 \end{smallmatrix}$ to any R_t (Fig. 9) and a dual spin to any pair $(t_{kl}, t_{k+1,l})$ or $(t_{kl}, t_{k,l-1})$ that gives a nonzero contribution:

$$\begin{aligned}
R_1 &\rightarrow \begin{smallmatrix} + & + \\ + & + \end{smallmatrix}, & R_2 &\rightarrow \begin{smallmatrix} - & - \\ - & - \end{smallmatrix}, & R_3 &\rightarrow \begin{smallmatrix} - & + \\ - & + \end{smallmatrix}, & R_4 &\rightarrow \begin{smallmatrix} + & - \\ + & - \end{smallmatrix}, \\
R_5 &\rightarrow \begin{smallmatrix} + & - \\ - & + \end{smallmatrix}, & R_6 &\rightarrow \begin{smallmatrix} - & + \\ + & - \end{smallmatrix}, & R_7 &\rightarrow \begin{smallmatrix} - & - \\ + & + \end{smallmatrix}, & R_8 &\rightarrow \begin{smallmatrix} + & + \\ - & - \end{smallmatrix}
\end{aligned} \tag{4.4a}$$

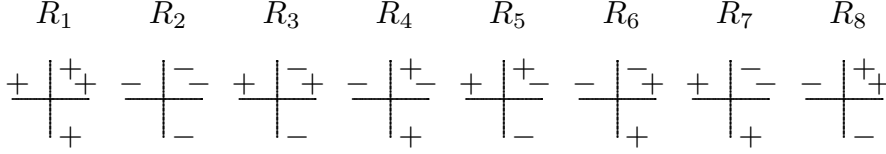


Fig. 9

for vertices,

$$\begin{aligned} (t, t') \rightarrow + & \quad \text{if} \quad t = 1, 4, 5, 8, & t' = 1, 4, 6, 7, \\ (t, t') \rightarrow - & \quad \text{if} \quad t = 2, 3, 6, 7, & t' = 2, 3, 5, 8 \end{aligned} \quad (4.4b)$$

for vertical links, and

$$\begin{aligned} (t, t') \rightarrow + & \quad \text{if} \quad t = 1, 3, 6, 8, & t' = 1, 3, 5, 7, \\ (t, t') \rightarrow - & \quad \text{if} \quad t = 2, 4, 5, 7, & t' = 2, 4, 6, 8 \end{aligned} \quad (4.4c)$$

for horizontal ones. Then the summation over dual spins is equivalent to the summation over $\{t_{kl}\}$, and this dual model corresponds to the local weights a_D, b_D, c_D, d_D . This is the Baxter's construction of duality.¹ Note that the dual lattice here coincides with the initial one and is not shifted by a half period as it is for the Ising model.

Now we shall extend this duality to correlation functions. Let us insert into the lattice several spin flop operators σ^1 . If, for example, one of them is inserted into the vertical link kl , the expression (4.3) turns into

$$\begin{aligned} \sum_{\mu_{kl}} (R_{t_{kl}})_{\mu_{k-1,l} \mu_{kl}}^{\mu_{kl} \nu_{k,l-1}} (R_{t_{k+1,l}})_{-\mu_{kl} \nu_{k+1,l}}^{\mu_{k+1,l} \nu_{k+1,l-1}} \\ = \pm \sum_{\mu_{kl}} (R_{t_{kl}})_{\mu_{k-1,l} \mu_{kl}}^{\mu_{kl} \nu_{k,l-1}} (R_{t_{k+1,l}})_{\mu_{kl} \nu_{k+1,l}}^{\mu_{k+1,l} \nu_{k+1,l-1}}, \end{aligned}$$

where the sign is '+' if there is no μ_{kl} in both $R_{t_{kl}}$ and $R_{t_{k+1,l}}$ and '-' if μ_{kl} is there. From Eq. (4.4b) we see that the sign coincides with the value of dual spin at this link. It means that a spin flop operator in the initial lattice corresponds to a spin variable in the dual lattice. Conversely, a spin variable in the initial lattice corresponds to a spin flop operator in the dual lattice. It means that there is a relation between correlation functions in the disordered and ferromagnetic regions

$$\begin{aligned} & \langle \mu_{k_1 l_1} \cdots \mu_{k_m l_m} \nu_{k'_1 l'_1} \cdots \nu_{k'_n l'_n} \sigma_{K_1 L_1}^1 \cdots \sigma_{K_M L_M}^1 \tau_{K'_1 L'_1}^1 \cdots \tau_{K'_N L'_N}^1 \rangle_D \\ &= \frac{1}{2} \sum_{i=0}^1 \langle \sigma_{k_1 l_1}^1 \cdots \sigma_{k_m l_m}^1 \tau_{k'_1 l'_1}^1 \cdots \tau_{k'_n l'_n}^1 \mu_{K_1 L_1} \cdots \mu_{K_M L_M} \nu_{K'_1 L'_1} \cdots \nu_{K'_N L'_N} \rangle_{F_1}^{(i)}, \end{aligned} \quad (4.5)$$

where σ_{kl}^1 and τ_{kl}^1 are spin flop operators σ^1 inserted into the vertical and horizontal links kl respectively. The summation over the boundary conditions $i = 0, 1$ is necessary, because the construction of duality needs summation over all configurations.

We now turn to parametrization. The transformation (4.1) leads to the parametrization

$$\begin{aligned}
a_D(\zeta) &= \mathcal{N} \rho(\zeta) \Theta_q(-\zeta) \Theta_q(-x/\zeta) \Theta_q(q^{\frac{1}{2}}\zeta) \Theta_q(q^{\frac{1}{2}}x/\zeta) / \Theta_q(-x) \Theta_q(q^{\frac{1}{2}}x), \\
b_D(\zeta) &= \mathcal{N} \rho(\zeta) \Theta_q(\zeta) \Theta_q(x/\zeta) \Theta_q(-q^{\frac{1}{2}}\zeta) \Theta_q(-q^{\frac{1}{2}}x/\zeta) / \Theta_q(-x) \Theta_q(q^{\frac{1}{2}}x), \\
c_D(\zeta) &= \mathcal{N} \rho(\zeta) \Theta_q(-\zeta) \Theta_q(x/\zeta) \Theta_q(q^{\frac{1}{2}}\zeta) \Theta_q(-q^{\frac{1}{2}}x/\zeta) / \Theta_q(x) \Theta_q(-q^{\frac{1}{2}}x), \\
d_D(\zeta) &= \mathcal{N} \rho(\zeta) \Theta_q(\zeta) \Theta_q(-x/\zeta) \Theta_q(-q^{\frac{1}{2}}\zeta) \Theta_q(q^{\frac{1}{2}}x/\zeta) / \Theta_q(x) \Theta_q(-q^{\frac{1}{2}}x), \\
\mathcal{N} &= \frac{1}{x} \frac{(q; q)_\infty (x^2; q)_\infty (qx^{-2}; q)_\infty}{(-1; q)_\infty (q^{\frac{1}{2}}; q)_\infty^2}.
\end{aligned} \tag{4.6}$$

The R -matrix $R_D(\zeta)$ satisfies Eqs. (2.4), (2.5a,b) and the same crossing symmetry equation as the ferromagnetic R -matrix

$$R_D(\zeta)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} = R_D(x/\zeta)_{\varepsilon_4 \varepsilon_1}^{\varepsilon_2 \varepsilon_3}. \tag{4.7}$$

So the CTMs are given by

$$\begin{aligned}
C_{NW}(\zeta) &= C_{SE}(\zeta) = \zeta^D, \\
C_{SW}(\zeta) &= C_{NE}(\zeta) = (x/\zeta)^D,
\end{aligned} \tag{4.8}$$

and the VO, $\Phi_\varepsilon(\zeta)$, satisfies the equations

$$\begin{aligned}
\sum_{\varepsilon'_1 \varepsilon'_2} R_D(\zeta_1/\zeta_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} \Phi_{\varepsilon'_1}(\zeta_1) \Phi_{\varepsilon'_2}(\zeta_2) &= \Phi_{\varepsilon_2}(\zeta_2) \Phi_{\varepsilon_1}(\zeta_1), \\
\xi^D \Phi_\varepsilon(\zeta) &= \Phi_\varepsilon(\zeta/\xi) \xi^D.
\end{aligned} \tag{4.9}$$

Therefore the VO correlation functions

$$F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n) = \text{Tr} \left(x^{2D} \Phi_{\varepsilon_1}(\zeta_1) \dots \Phi_{\varepsilon_n}(\zeta_n) \right), \quad n = 0, 1, 2, \dots, \tag{4.10}$$

satisfy the equations

$$\begin{aligned}
F_{\varepsilon_1 \dots \varepsilon_n}(\xi \zeta_1, \dots, \xi \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n) \\
\sum_{\varepsilon'_j \varepsilon'_{j+1}} R_D(\zeta_j/\zeta_{j+1})_{\varepsilon_j \varepsilon_{j+1}}^{\varepsilon'_j \varepsilon'_{j+1}} F_{\dots \varepsilon'_j \varepsilon'_{j+1} \dots}(\dots, \zeta_j, \zeta_{j+1}, \dots) &= F_{\dots \varepsilon_{j+1} \varepsilon_j \dots}(\dots, \zeta_{j+1}, \zeta_j, \dots), \\
F_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}(x^2 \zeta_1, \zeta_2, \dots, \zeta_n) &= F_{\varepsilon_2 \dots \varepsilon_n \varepsilon_1}(\zeta_2, \dots, \zeta_n, \zeta_1), \\
\sum_{\varepsilon} F_{\varepsilon \varepsilon \varepsilon_1 \dots \varepsilon_n}(x \zeta, \zeta, \zeta_1, \dots, \zeta_n) &= F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n), \\
F() &= 1, \quad F_\varepsilon(\zeta) = \frac{1}{2}
\end{aligned} \tag{4.11}$$

with Eq. (3.10) for probabilities.

Let us relate this construction to the ferromagnetic case. Let

$$\Phi_\varepsilon(\zeta) = \Phi'_+(\zeta) + \varepsilon \Phi'_-(\zeta). \quad (4.12)$$

It is easy to check using Eq. (4.1) or (4.2) that

$$\sum_{\varepsilon'_1 \varepsilon'_2} R_F(\zeta_1/\zeta_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} \Phi'_{\varepsilon'_1}(\zeta_1) \Phi'_{\varepsilon'_2}(\zeta_2) = \Phi'_{\varepsilon_2}(\zeta_2) \Phi'_{\varepsilon_1}(\zeta_1),$$

Therefore $\Phi'_\varepsilon(\zeta)$ is the ferromagnetic VO. More precisely,

$$\Phi'_\varepsilon(\zeta) \sim (\Phi_\varepsilon^{(0)}(\zeta) + \Phi_\varepsilon^{(1)}(\zeta))_{\text{ferromag.}} \quad (4.13)$$

This imposes an additional condition on the VO correlation functions

$$F_{\varepsilon_1 \dots \varepsilon_n}(\zeta_1, \dots, \zeta_n) = F_{-\varepsilon_1 \dots -\varepsilon_n}(\zeta_1, \dots, \zeta_n). \quad (4.14)$$

On the other hand it is straightforward to obtain Eq. (4.5) from (4.10)–(4.14).

We ought to make a remark. We have implicitly supposed that $a, b, c, d > 0$. But to cover the whole region $\frac{1}{2}(a_D + b_D + c_D + d_D) > a_D, b_D, c_D, d_D > 0$ we need to consider negative values of a, b, c, d . In this case the applicability of the above construction can be justified by the fact that there are no phase transitions in the disordered region. Hence, the partition function and correlation functions may be analytically continued from the subregion $a, b, c, d > 0$.¹

5. Six-Vortex Model in Critical Region

One of the most interesting problems is to obtain a description of the six-vortex model in the critical region which corresponds to the disordered region of the eight-vortex model with $d = 0$. We could set $d_D(\zeta) = 0$, but it is not convenient, because the crossing transformation (4.7) does not preserve this condition. Instead, we apply the transformation described in Sec. 3 to the disordered region

$$a'_D(\zeta) = c_D(\zeta), \quad b'_D(\zeta) = d_D(\zeta), \quad c'_D(\zeta) = a_D(\zeta), \quad d'_D(\zeta) = b_D(\zeta). \quad (5.1)$$

This transformation maps the disordered region onto itself. The crossing symmetry of the form (2.5c)

$$a'_D(x/\zeta) = b'_D(\zeta), \quad c'_D(x/\zeta) = c'_D(\zeta), \quad d'_D(x/\zeta) = d'_D(\zeta)$$

preserves the condition $d'_D(\zeta) = 0$. On the other hand, the equations for VO correlation functions and probabilities are given by (2.12), (2.11) for these weights, but without superscripts $^{(i)}, {}^{(1-i)}$, with the substitution $R(\zeta) \rightarrow R'_D(\zeta)$, and with additional condition (4.14).

It is convenient to use here an additive spectral parameter instead of multiplicative one. Change variables

$$\zeta = e^{-\pi u'/2I}, \quad x = e^{-\pi \lambda'/2I}. \quad (5.2)$$

It is well known¹ that

$$a : b : c : d = \sinh(\lambda' - u') : \sinh u' : \sinh \lambda' : k \sinh \lambda' \sinh u' \sinh(\lambda' - u'),$$

where $\sinh z = -i \operatorname{sn} iz$, and k is the module of the elliptic functions. Consider the limit $q^{\frac{1}{2}} \rightarrow -1$ ($k \rightarrow -1$). Then

$$a : b : c : d = \tan(\lambda' - u') : \tan u' : \tan \lambda' : -\tan \lambda' \tan u' \tan(\lambda' - u').$$

In this limit $a + b - c - d = 0$ or $d'_D = 0$. Introducing new variables

$$\begin{aligned} u &= 2u', & \lambda &= 2\lambda', \\ 0 &< u < \lambda < \pi, \end{aligned} \quad (5.3)$$

we easily obtain

$$\begin{aligned} a'_D(u) &= \rho'(u) \sin(\lambda - u), \\ b'_D(u) &= \rho'(u) \sin u, \\ c'_D(u) &= \rho'(u) \sin \lambda, \\ d'_D(u) &= 0, \end{aligned} \quad (5.4)$$

where the function $\rho'(u)$ providing initial condition, unitarity, and crossing symmetry can be adopted from Ref. 9:

$$\begin{aligned} \rho'(u) &= \frac{1}{\pi} \Gamma\left(\frac{\lambda}{\pi}\right) \Gamma\left(1 - \frac{\lambda}{\pi}\right) \Gamma\left(1 - \frac{\lambda - u}{\pi}\right) \prod_{p=1}^{\infty} \frac{r_p(u) r_p(\lambda - u)}{r_p(0) r_p(\lambda)}, \\ r_p(u) &= \frac{\Gamma\left(\frac{2p\lambda - u}{\pi}\right) \Gamma\left(1 + \frac{2p\lambda - u}{\pi}\right)}{\Gamma\left(\frac{(2p+1)\lambda - u}{\pi}\right) \Gamma\left(1 + \frac{(2p-1)\lambda - u}{\pi}\right)}. \end{aligned} \quad (5.5)$$

Now the equations for correlation functions in the critical region of the six-vertex model are evident.

6. Conclusion

We extended the vertex operator formalism from the antiferromagnetic region to the ferromagnetic and disordered ones. The crucial role belongs to the parametrization of the weights and to the crossing symmetry equation which are different in different regions. Baxter's symmetries prove the correctness of the result.

In this paper we did not consider vertex operators of type II. These operators are very important because they diagonalize explicitly the transfer matrix and give the spectrum of excitations. Evidently, they can be introduced in every region through the connection with the elliptic algebra $\mathcal{A}_{q,-x}(\widehat{sl}_2)$.⁸

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Appendix

In practical calculations the relations between different parametrizations may be useful. Here we cite them:

$$\begin{aligned} R_F(\zeta|q, x) &= R(\zeta|q, x_F)|_{\rho=\rho_F}, \\ R_D(\zeta|q, x) &= R(\zeta_D|q_D, x_D)|_{\rho=\rho_D}. \end{aligned} \quad (A.1)$$

Using the usual techniques of elliptic functions we obtain

$$\begin{aligned} x_F &= -q^{-\frac{1}{2}}x, \\ \rho_F(\zeta|q, x_F) &= -\frac{q\zeta}{x_F^2}\rho(\zeta|q, -q^{\frac{1}{2}}x), \\ \frac{1}{\log q_D} &= \frac{1}{\log q} + \frac{i}{2\pi}, \quad q'_D = iq', \quad \zeta_D = \zeta^\alpha, \quad x_D = (q^{\frac{1}{2}}x)^\alpha, \quad \alpha = \frac{\log q_D}{\log q}, \\ \frac{\rho_D(\zeta_D|q_D, x_D)}{\rho(\zeta|q, x)} &= \frac{1}{\mathcal{N}} \frac{q_D^{\frac{1}{2}}}{x_D^2} \frac{\Theta_{q_D^2}(x_D^2)\Theta_{q_D^2}(\zeta_D^2)\Theta_{q_D^2}(x_D^2/\zeta_D^2)\Theta_q(x)\Theta_q(-q^{\frac{1}{2}}x)}{\Theta_q(\zeta)\Theta_q(-x/\zeta)\Theta_q(-q^{\frac{1}{2}}\zeta)\Theta_q(q^{\frac{1}{2}}x/\zeta)}. \end{aligned} \quad (A.2)$$

Here the relation between the conjugate parameters q' and q'_D is cited.

Note that the ferromagnetic spontaneous polarization can be written in the form

$$\langle \varepsilon \rangle = \frac{(qx_F^2; qx_F^2)_\infty^2 (-q; q)_\infty^2}{(-qx_F^2; qx_F^2)_\infty^2 (q; q)_\infty^2}.$$

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